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THE EQUILIBRIUM OF A HEAVY HOMOGENEOUS CHAIN IN A UNIFORMLY ROTATING PLANE

BY EDWIN BIDWELL WILSON

1. Introduction. The problem of the equilibrium of a chain has attracted considerable attention since the days when James Bernoulli first solved the simple case of a uniform chain hanging by both ends in the constant field of gravity. A large amount of theory has been developed * and a great variety of problems has been solved. It appears, however, that for the most part the cases which have yielded integrable differential equations belong to two types: Those of equilibrium in a parallel field or in a central field.† Both these fields are rectilinear, so that the question of the equilibrium of a chain with one end free is solved intuitively. The first plane curvilinear field to occur to one would probably be that of a constant force parallel to a line combined with a force at right angles to the line and varying directly as the distance from the line. Furthermore precisely such a field is afforded by the idealization of a common physical phenomenon.

Consider a uniform chain or string suspended by one end in the field of gravity and constrained to rotate by twisting that end. It is well known that the chain will either swing entirely to one side of the vertical or will take the form of a series of arches which increase in height from the fixed toward the free end. The motion of the chain rapidly approaches a steady state in which each point is revolving at a constant speed in a horizontal circle concentric with the vertical axis through the point of support. The effects of torsion and of friction against the air will somewhat disturb the experiment and very greatly complicate the differential equations which determine the figure of the curve. If we idealize the problem by omitting these disturbing effects and by assuming the chain to be uniform and mathematically perfect, that is, of no appreciable thickness and quite flexible and inelastic, we have a problem easy to state in differential equations. It is this problem with some generalization which will be discussed in the following pages.

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* Perhaps the best account of the theory which is readily accessible is that given by Appell in his *Traité de mécanique rationnelle*, volume I, chapter 6, pages 180–204. A shorter treatment is found in Gray's *Treatise on Physics*, volume I, chapter 8, pages 286–295.

† For numerous problems see Appell, loc. cit., or Walton's *Problems in Theoretical Mechanics*, third edition, pages 114–143.

According to a fundamental principle of dynamics, the uniform rotation of a system may be done away if the corresponding centrifugal force be introduced. As the centrifugal force varies directly with the distance from the axis of rotation, the problem of the equilibrium of a rotating chain reduces to that of equilibrium in the curvilinear field above mentioned, provided only that the chain lies entirely in one plane. Whether or not this condition be fulfilled depends on the initial conditions. The equations of equilibrium are

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right)+\rho X=0,\quad \frac{d}{ds}\left(T\frac{dy}{ds}\right)+\rho Y=0,\quad \frac{d}{ds}\left(T\frac{dz}{ds}\right)+\rho Z=0,$$

where T is the tension, ρ the density, and X, Y, Z the external forces per unit mass at any point of the chain.

If the x-axis be vertical and directed upwards, the equations for the present problem become

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right)-\rho g=0,\quad \frac{d}{ds}\left(T\frac{dy}{ds}\right)+\rho\omega^2 y=0,\quad \frac{d}{ds}\left(T\frac{dz}{ds}\right)+\rho\omega^2 z=0,$$

where ω is the angular velocity. It will be worth while to show that if the chain is attached at one point to the axis of rotation, it must lie entirely in a vertical plane through the axis. For this purpose the cylindrical equations of the chain are useful. These may be deduced exactly as the Cartesian equations are usually deduced, i. e., from the formulas

$$d(T\alpha)+\rho F_1 ds=0,\quad d(T\beta)+\rho F_2 ds=0,\quad d(T\gamma)+\rho F_3 ds=0,$$

where F_1, F_2, F_3 represent the forces along any three mutually perpendicular directions and α, β, γ are the direction cosines of the element of arc with reference to those directions. If the directions are X and, in the yz-plane, the radial direction R and the perpendicular to R, we have

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right)+\rho X=0,\quad \frac{d}{ds}\left(T\frac{dr}{ds}\right)+\rho R=0,\quad \frac{d}{ds}\left(T\frac{r\,d\theta}{ds}\right)+\rho\Theta=0,$$

where X, R, Θ now represent the components of the force. In the case in hand $X=-g$, $R=\omega^2 r$, $\Theta=0$. Hence $T\,r d\theta/ds=$ const., and if the chain has a point on the axis of rotation, the constant vanishes. Hence Θ is constant and the chain must lie in a vertical plane through the axis. Hereafter, as stated in the title, we shall restrict the discussion to this case where the chain lies in a plane.

2. The differential equations of the problem. Let the chain lie in the xy-plane and assume that the x-axis is vertical with the positive end up. The equations of the chain then reduce to

$$(1) \qquad \frac{d}{ds}\left(T\frac{dx}{ds}\right) - \rho g = 0, \qquad \frac{d}{ds}\left(T\frac{dy}{ds}\right) + \rho\omega^2 y = 0$$

in rectangular coordinates. The intrinsic equations are

$$(2) \qquad \frac{dT}{ds} + \rho S = 0, \qquad T + \rho R N = 0,$$

where S and N are the tangential and normal components of the applied forces and R the radius of curvature of the chain. In the case in hand the forces are derivable from the potential.

$$V = gx - \tfrac{1}{2}\omega^2 y^2, \qquad S = -\,\partial V/\partial s.$$

Hence the first equation of (2) may be integrated to

$$(3) \qquad T - \rho V = T - \rho g x + \tfrac{1}{2}\rho\omega^2 y^2 = \text{const.}$$

If T_0 be the tension in the chain at the origin $(0, 0)$, we have

$$(3') \qquad T = T_0 + \rho g x - \tfrac{1}{2}\rho\omega^2 y^2.$$

Inasmuch as T cannot become negative, it is apparent that the only region of the plane in which the chain may lie is the interior of the parabola

$$(4) \qquad T = T_0 + \rho g x - \tfrac{1}{2}\rho\omega^2 y^2 = 0,$$

which has its vertex downwards and situated at the point $x = -\,T_0/\rho g$.

The second of the intrinsic equations (2) may be used to obtain the differential equation of the second order for the chain. For

$$N = g \sin\tau + \omega^2 y \cos\tau,$$

where τ is the inclination of the curve to the x-axis. Hence

$$T + \rho(g\sin\tau + \omega^2 y\cos\tau)\,\frac{(1 + y'^2)^{\frac{3}{2}}}{y''} = 0.$$

Or

$$(5)\qquad y'' = -\frac{\rho\omega^2 y + \rho g y'}{T_0 + \rho g x - \frac{1}{2}\rho\omega^2 y^2}\,(1 + y'^2).$$

It should be noted that this equation contains the quantity T_0 which is unknown: for with a given chain of length l suspended either at both ends or only at one, the tension at every point is a consequence of the position of the chain and is not to be counted among the initial conditions. In fact the tension at the point of support is a function of l. The differential equation of the chain is really of the third order and may be found by eliminating T_0 from (5) and that equation differentiated with respect to x. The form containing T_0 will, however, answer our purposes.*

Another method of treating the problem is by the calculus of variations. The condition for equilibrium is that the potential energy be an extremum — a minimum if the equilibrium is to be stable. As the length of the chain is constant the problem is one in relative minima; in fact we have

$$\int V ds \quad \text{a minimum}, \qquad \int ds \quad \text{constant}.$$

Hence it is necessary to minimize the integral

$$(6)\qquad I = \int (\lambda g + gx - \tfrac{1}{2}\omega^2 y^2)\, ds,$$

where λ is a parameter. By a theorem due to Weierstrass,† an integral of

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* Equation (5) can be integrated once by introducing the arc s and its derivative ds/dx. The integral may be found more readily by operating directly on the fundamental equations (1). The result is, if the point of support of the chain be at the origin,

$$(\rho g s + T_0 \cos\tau_0)\, ds/dx = T_0 + \rho g x - \tfrac{1}{2}\rho\omega^2 y^2,$$

where τ_0 is the inclination of the string at the origin. The derivation of this result, which is equivalent to our (7) below, will be left to the reader who may thus entirely avoid the use of the calculus of variations in arriving at this important equation (7). It should be noted here again that τ_0 is not properly to be considered among the initial conditions any more than T_0; it is indeed a function of l.

† Instead of reverting to Weierstrass's treatment, we may note that integrals of the form (6) are particularly simple to treat by methods recently developed by Bliss in the *Transactions of the American Mathematical Society*, volume 8 (1907), pages 405–414. He considers the standard form of the integrand in the calculus of variations as $F(x, y, \tau)ds$. This is highly convenient for the present problem: for here F does not contain τ explicitly and moreover the extremals may become perpendicular to the x-axis which is not allowed in case the integrand has the form $F(x, y, y')dx$. Bliss obtains the necessary condition $F + F_{\tau\tau} \geqq 0$ for a minimum — a condition which in this case restricts (x, y) to the interior or contour of the parabola (4′).

this form cannot be a minimum if the coefficient of ds becomes negative. This shows that the only possible region for minimizing extremals is the interior of the parabola

$$\lambda g + gx - \tfrac{1}{2}\omega^2 y^2 = 0. \tag{4'}$$

Hence it appears, as may readily be shown also by deriving the differential equation of the extremals, that λ is to be identified with $T_0/\rho g$, and the fact that T_0 is not to be regarded as among the initial conditions but to be determined from the length l of the chain is further emphasized.

Some interesting and important results may be obtained by applying the ordinary rules of variation directly to (6).* For this purpose we may indicate the limits by 0 and 1 and consider them as variable.

$$\delta I = \int_0^1 \left\{ (g\delta x - \omega^2 y \delta y) ds + (\lambda g + gx - \tfrac{1}{2}\omega^2 y^2) \frac{dx\,\delta dx + dy\,\delta dy}{ds} \right\}.$$

By integration by parts we have

$$\delta I = \int_0^1 \left\{ g ds - d\left[(\lambda g + gx - \tfrac{1}{2}\omega^2 y^2) \frac{dx}{ds} \right] \right\} \delta x$$

$$+ \int_0^1 \left\{ - \omega^2 y ds - d\left[\lambda g + gx - \tfrac{1}{2}\omega^2 y^2) \frac{dy}{ds} \right] \right\} \delta y$$

$$+ \left[(\lambda g + gx - \tfrac{1}{2}\omega^2 y^2) \frac{dx\,\delta x + dy\,\delta y}{ds} \right]_0^1 .$$

The brace under either integral may be set equal to zero to obtain the extremals. The first of these may be integrated and gives

$$gs + \text{const} = (\lambda g + gx - \tfrac{1}{2}\omega^2 y^2) \frac{dx}{ds} \cdot \tag{7}$$

The conditions at the limits are

$$[(\lambda g + gx - \tfrac{1}{2}\omega^2 y^2)(dx\,\delta x + dy\,\delta y)]_0^1 = 0,$$

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* This method of procedure which is almost always the most convenient in practice is admirably treated by de la Vallée Poussin in his *Cours d'analyse infinitésimale*, volume 2, pages 325–346.

and show that: *If the extremities of the chain are movable on fixed curves, the chain must be perpendicular to those curves.**

3. General discussion of the equilibrium. As equation (5) appears not to be integrable except into (7) which contains the arc and its derivative and is probably not further integrable, it will be well to begin with a general discussion of the equilibrium. In the first place we may state the result: *Two chains precisely alike except for different densities will have the same positions of equilibrium.* This may be seen at once from the equations (1) or (2). The physical reasoning which may be employed to reach the same conclusion is equally simple. It is merely necessary to cite the principle that if all the applied forces in a statical system be reduced in the same ratio, the system remains in equilibrium and the forces developed, that is, frictional forces, reactions at fixed constraints, and so on, are reduced in the same ratio. For similar reasons we may state: *The position of equilibrium will be the same in all fields which have the same value for the ratio* $\omega^2 : g$. In fact until the unit of time, which is of no consequence in statics, has been specified, it is only the ratio of ω^2 and g which can be said to be known. Therefore without loss of generality the equations may be simplified by setting $\rho = 1$ and $\omega^2/g = \kappa$.

The case most easily realized in practice is where the upper end of the string is fixed at a point in the axis of rotation and the other end allowed to hang free: the origin may then be taken at the point of support. It is well known in the theory of the conical pendulum that the angular velocity must attain a certain value relative to the length of the pendulum before the position of equilibrium will depart from the vertical. In case the pendulum is a uniform rod of length l, the total moment of the centrifugal forces about the point of support is $\frac{1}{3}\rho\omega^2 l^3 \sin\theta \cos\theta$, where θ is the angle of inclination. The moment of the gravitational forces is $\frac{1}{2}\rho g l^2 \sin\theta$. Hence for equilibrium

$$\sec\theta = \tfrac{2}{3}\omega^2 l/g; \qquad \text{and} \qquad \omega^2 l/g = \kappa l > \tfrac{3}{2}$$

is the necessary and sufficient condition that the position of stable equilibrium is not vertical.

It is readily shown that: *The chain will depart from the vertical if* $\kappa l > \frac{3}{2}$.

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* The possible exception of a free end on (4′), which would satisfy the condition identically, proves later not to be an exception.

For if $\kappa l > \frac{3}{2}$, the facts concerning the rod show that there are positions in the neighborhood of the vertical, namely, all straight lines slightly inclined, for which $\int V\,ds$ is less than its value along the vertical subject to the condition of constant length. Hence there are variations of the integral which are negative, weak variations at that, and the vertical cannot be a position of stable equilibrium. Similarly in case the chain is attached to two points of the vertical axis separated by a distance $a < l$, it may be shown that the position of equilibrium will depart from the vertical if $\kappa(l + a) > 3$. Whether on the other hand the chain will hang in the vertical if these inequalities are not satisfied is a question to be taken up later.

If the chain has a free end the tension at that point is zero and conversely if the tension at any point is zero the chain may be considered as cut and having a free end at that point. As the introduction of new constraints never disturbs an existing equilibrium, the chain may be considered as pinned in the field indefinitely near the free end, and hence it is physically obvious that: *At the free end a chain must have the direction of the field, that is, must be perpendicular to the equipotential* (4′). Again if the tension be not zero, the second of the intrinsic equations (2) shows that: *A chain has a point of inflexion where and only where the chain has the direction of the field (except perhaps at the free end, if one exists).* The curvature at the free end requires special investigation because here (2) gives $R = 0/0$.

To treat this question, transform (5) to a new origin of x so that the y-axis passes through the free end upon the parabola (4′) at a point $y = y_0$, and perform the integration which gives (7). Let s be measured positively from the free end up. The constant in (7) vanishes because s and the first member of (4′) vanish, and $\lambda = \frac{1}{2}\kappa y_0^2$. We have then

$$s\,\frac{ds}{dx} = x + \frac{1}{2}\,\kappa(y_0^2 - y^2). \tag{7′}$$

If we plot an element of the tangent $y = y_0(1 - \kappa\Delta x)$, we obtain at the end of this lineal element

$$\frac{ds}{dx} = \frac{\Delta x - \frac{1}{2}\kappa[y_0^2 - y_0^2(1 - \kappa\Delta x)^2]}{\sqrt{1 + \kappa^2 y_0^2}\quad \Delta x}.$$

Hence

$$\Delta\,\frac{ds}{dx} = \sqrt{1 + \kappa^2 y_0^2} - \frac{ds}{dx} = \frac{-\frac{1}{2}\,\kappa^3 y_0^2\Delta x}{\sqrt{1 + \kappa^2 y_0^2}}.$$

Hence if we assume that as we plot the curve by the Cauchy-Lipschitz method of approximation, the limit approached by the polygon is the integral of the differential equation, and that the curvature of the integral may be obtained from the polygon,* we find

$$\frac{d^2s}{dx^2} = -\frac{1}{2}\frac{\kappa^3 y_0^2}{\sqrt{1+\kappa^2 y_0^2}}.$$

Inasmuch as

$$\frac{d^2s}{dx^2} = \frac{d\sec\tau}{dx} = \tan\tau\sec\tau\,\frac{d\tau}{ds}\cdot\frac{ds}{dx} = -\frac{\kappa y_0(1+\kappa^2 y_0^2)}{R},$$

where R is the radius of curvature, it follows that

$$R = \frac{2(1+\kappa^2 y_0^2)^{\frac{3}{2}}}{\kappa^2 y_0^2}.$$

And if the radius of curvature of the lines of force

$$y = y_0 e^{-\kappa x}$$

defined by the differential equation $dy/dx = -\omega^2 y/g = -\kappa y$ be computed, it is found to be one-half of the value of R above. Hence: *The radius of curvature of the chain at its free end is neither zero nor infinite but twice the radius of curvature of the field at that point.*

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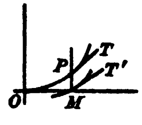
* It is well to note precisely what this assumption amounts to. In the figure the curve OP represents the integral of the differential equation. The first step in the Cauchy-Lipschitz method is to construct the tangent OM. At the point M the tangent MT' is drawn. This is not tangent to the integral curve OP but to the neighboring integral curve through M. Thus the first two lines in the approximation are OM and MT'. The curvature of the integral OP is the limit of the inclination of PT divided by the arc OP as $P \doteq O$. In the text this limit has been replaced by the limit of the inclination of MT' divided by OM as $M \doteq O$. If now it be assumed, as it is, that at the point O, the curvature of the integral curve OP is not infinite, the difference between OP and OM is an infinitesimal of the third order at least, and PM is at least of the second order. The difference between the inclinations of PT and MT' will then also be of the second order at least Consequently in the limit OP may be replaced by OM and the inclination of PT may be replaced by that of MT'. Hence under the assumption that the curvature of OP at O is finite, the statement in the text is justified and the curvature may be found by the process indicated.



Suppose that the chain, whether it has a free end or not, lies in a curve which has maxima and minima of y. For brevity maxima and minima may be designated as extreme points. At each extreme point the tension is parallel to the x-axis, and as the tension is null at the free end (if there is one) we may include this point among the extreme points. As the chain is in equilibrium, it may be treated for the moment as a rigid body. The forces acting between two extreme points are the tensions at those points and the force of gravity parallel to the axis, and the elemental centrifugal forces perpendicular to the axis. These must form a system in equilibrium. Hence the centrifugal forces balance. But they have the form $\omega^2 y ds$ and hence between extreme points,

$$\int \omega^2 y\, ds = 0.$$

This is precisely the condition that the center of gravity of the chain shall be on the x-axis. We may therefore state: *The center of gravity of the arc of the chain between any two extreme points lies on the axis.* From this result some corollaries: 1° *The chain must cross the axis between successive extreme points.* 2° *The free end of the chain cannot lie on the axis.*

4. The form of the curves of equilibrium. To obtain a more precise idea of the curves of equilibrium, it is necessary to resort to a detailed discussion of the differential equation which defines them. It has been seen that the inflexions of the curves are tangent to the lines of force. Let us show that: *No inflexion can lie upon the axis.* Take the origin at the supposed inflexion. Equation (5) shows that if $y'' = 0$ and $y = 0$, then $y' = 0$. Now the axis of rotation is known to be a possible solution of the equation (5) of the second order with the initial conditions $y' = y = x = 0$. Hence, relying on the uniqueness of the solution, we see that no other solution with these initial conditions is possible, and therefore no inflexion can lie on the axis except in the case where the whole chain lies along the axis. The same conclusion may be derived directly by computing, by successive differentiations of (5), the values of all the higher derivatives $y^{(n)}$ at the point (0, 0) and noting that they all vanish.

Next consider the particular case of the chain with a free end and take (7′) as the defining equation. If this be written as

$$\frac{ds}{dx} = \frac{x + \frac{1}{2}(y_0^2 - y^2)}{s},$$

it appears that ds/dx cannot be infinite except at $s = 0$. When $s = 0$, however, it has been seen that ds/dx is the finite quantity $\sqrt{1 + \kappa^2 y^2}$. Hence: *In case the chain has a free end, the slope of the curve is everywhere finite.* Consider

$$\frac{ds}{dx} - 1 = \frac{x - s + \frac{1}{2}\kappa(y_0^2 - y^2)}{s}.$$

As $1 \leqq \sec \tau$, the numerator is always positive or zero. At the maximum and minimum points it vanishes. These points are therefore given by the equation

$$x - s + \tfrac{1}{2}\kappa(y_0^2 - y^2) = 0.$$

Here $x - s$ is zero at the free end and becomes increasingly negative. So at successive extreme points $y_0^2 - y^2$ must become increasingly positive, that is: *Each successive extreme point from the free end up is nearer the axis than the preceding.* Similarly it may be seen that: *At each successive intersection of the curve with the axis the inclination of the curve to the axis is less and is approaching zero as its limit.*

To return to the question of inflexions, we may note that it has not yet been shown that the chain actually changes its concavity at the inflexions. We shall therefore prove that: *The points of inflexion are of the first order, that is, $f'''(x) \neq 0$.* On differentiating (5) we find

$$y''' = -\frac{T(\kappa y' + y'')(1 + y'^2) + T(\kappa y + y')2yy'' - (\kappa y + y')\,dT/dx}{T^2}.$$

As $dT/ds = -S$ and is finite, dT/dx is finite and y''' reduces to $-\kappa y'(1 + y'^2)/T$ at an inflexion and cannot vanish. We may next show that: *The curve must change its concavity between an extreme point and the next intersection with the axis.* For consider a minimum. As the curve is traced from this point up, the slope is positive and remains positive to the following maximum. The curve must therefore change its concavity while its slope is positive. As the slope at an inflexion is $-\kappa y$, the ordinate of the inflexion is negative. A maximum may be treated in the same manner. It may next be shown that: *There is only one inflexion between successive extreme points.* The easiest way to see this is to start at an inflexion, say on the positive side of the axis, and trace the curve backward toward the free end. At the inflexion the slope is negative and becomes numerically less as the curve rises toward the maximum. But as the ordinates increase the slope of the field becomes negatively greater. Hence the maximum is reached before another inflexion. The slope of the curve then becomes positive and consequently cannot coincide with the slope of the field until the axis is crossed. When an inflexion on the negative side of the axis is reached the slope decreases as the curve rises whereas the slope of the field increases. Hence the minimum is reached before another inflexion, and so on.

It is readily shown that: *The maximum slope in each arch of the curve approaches zero as its limit* as well as the slope at the intersection with the axis. Hence: *The inflexions must approach the axis as a limit* although none of them lie upon it. There is no difficulty in seeing that: *The inclination of the curve is nowhere so great as at the free end and consequently the tension in the chain must be constantly increasing.* If we assume, as will be justified in the next section, that the ordinates of the curve approach zero as their limit, it appears that: *At least qualitatively the shape of the curve is represented by the graph of the Bessel's function* J_0. This will be further exemplified in article 5.

If the chain has not a free end, suppose that the y-axis be taken through the lowest intersection of the curve with the axis of x. Then equation (7) takes the form,

$$(s + \lambda \cos \tau_0) \frac{ds}{dx} = \lambda + x - \tfrac{1}{2}\kappa y^2,$$

where τ_0 is the inclination at the origin. According to the convention established above, s is to be measured positively if the curve starts off into either of the quadrants above the y-axis; in this case the discussion runs about as for the case of a free end. But if the curve starts out in either of the other two quadrants s has negative values. The form of the differential equation (5) is such that the curve cannot have any singular points until it meets the parabola (4′). In this case it cannot meet the parabola as there is no free end. We may therefore trace the curve back through negative values of s to a point where $s + \lambda \cos \tau_0$ vanishes and the curve becomes perpendicular to the axis. Obviously: *There is only one point at which the curve may have a tangent per-*

pendicular to the axis and this is situated below the lowest intersection with the axis, except in the limiting case where the tangent at the lowest point is itself perpendicular to the axis. If the y-axis be moved into coincidence with this tangent, the equation of the curve becomes

$$s\frac{ds}{dx} = x + \tfrac{1}{2}\kappa(y_0^2 - y^2),$$

which is identical in form with the equation (7′), in the case of a free end, except that here y_0 being the point of the parabola above the origin cannot be the ordinate of the point with the vertical tangent but must be greater. It is obvious by reasoning similar to that employed in the former case that: *The curve cannot rise so high as $y = y_0$. The ordinate will be doubly valued for all positive values of x and except for the vicinity of the vertical tangent each of the branches will behave qualitatively like the former curve.* Further details may be left to the reader.

5. Approximations in the differential equation. Let us next plot the curves by following successive tangent lines. If the curve has a free end, we may start at this end with equation (7′). This method avoids having to deal with the question of the value of T_0 or λ. In fact the length of the chain will be a consequence of the position of equilibrium from this point of view, instead of the reverse. To simplify matters and to have a definite parabola let us assume $\kappa = 2$. It may be noted that: *The constant κ is the reciprocal of the semi-latus rectum. The inequality $\kappa l > \frac{3}{2}$, if we may assume the necessity as well as sufficiency of the condition, shows that the least length of the chain is 3/2 the semi-latus rectum, that is, 3/2 the radius of curvature at the vertex of the parabola.* From the fact that the qualitative discussion has shown that the curve does not change its concavity between the free end and its first root it follows that: *No matter how near the vertex of the parabola the the free end may be, the length of the chain must be at least equal to the radius of curvature.* As there seems to be no other way of showing that this length must be half as great again as this radius of curvature, except by investigating the roots of the solution of the differential equation (a rather difficult and inaccessible problem), it may be interesting to plot one curve which starts very near the vertex.

Let us assume $y_0 = 0.05$, then at the start $y' = 0.1$. The following table of values may then be constructed.

Δs	s	x	y	$\log (ds/dx)$	$\tau = \sec^{-1} \frac{ds}{dx}$	$\Delta\tau$
0.	0.	0.	+0.05	0.0 021 607	5 42′ 38″	
0.05	0.05	0.04 975 165	0.04 502 485	0.0 019 449	5 25 6	17′ 32″
0.05	0.10	0.09 952 823	0.04 030 351	0.0 017 503	5 8 26	16 40
0.05	0.15	0.14 932 711	0.03 582 355	0.0 015 715	4 52 16	16 10
0.05	0.20	0.19 914 651	0.03 157 770	0.0 014 078	4 36 39	15 37
0.05	0.25	0.24 898 469	0.02 755 943	0.0 012 580	4 21 32	15 7
0.05	0.30	0.29 884 008	0.02 375 825	0.0 011 214	4 6 56	14 36
0.05	0.35	0.34 871 114	0.02 016 972	0.0 009 968	3 52 50	14 6
0.05	0.40	0.39 859 651	0.01 678 587	0.0 008 834	3 39 11	13 39
0.05	0.45	0.44 849 491	0.01 360 002	0.0 007 810	3 26 7	13 4
0.05	0.50	0.49 840 507	0.01 678 587	0.0 006 880	3 13 27	12 40
0.05	0.55	0.54 832 592	0.00 779 195	0.0 006 037	3 1 13	12 14
0.05	0.60	0.59 825 647	0.00 515 748	0.0 005 279	2 49 24	11 45
0.05	0.65	0.64 819 573	0.00 269 369	0.0 004 597	2 38 9	11 19
0.05	0.70	0.69 814 284	+0.00 039 431	0.0 003 985	2 27 15	10 54
0.05	0.75	0.74 809 698	−0.00 174 670			

The object of giving this table in extenso is to show the large number of figures necessary to get anything like a trustworthy result. As it is, the seconds are by no means determined; the differences of the log-secant are only about one unit in the last place given. The point at which the curve crosses the axis is found from the table to be $s = 0.718+$. Under our assumption that $\kappa = 2$, this length should exceed (probably by only an insignificant amount) the minimum length 0.75. As a seven-place table did not afford a ready accuracy sufficient to make the calculations when the increments of arc are taken smaller and as such a calculation would be very tedious in any case, the calculation for increments of arc equal to 0.10 instead of 0.05 was carried through. This gave a length $s = 0.695-$, which is less than the other as was to be expected. Moreover the difference 0.023 or 0.024 between the two results is of such magnitude as to render it quite plausible that if the approximation were pushed to the limit the length would slightly exceed 0.75. It is indeed not difficult to obtain roughly an idea of the order of magnitude of the error introduced at each step of the approximation. If R be the radius of curvature and $\Delta\tau$ the angle between successive tangents, the departure of the tangent from the curve is $R(\Delta\tau)^2/2$. As $R = \Delta s/\Delta\tau$, this becomes $\Delta\tau\,\Delta s/2$. In the present problem $\Delta\tau$ is on the average about 1/250th of a radian. The individual errors in the fifteen approximations are cumulative, and hence the total error in the ordinate must be expected to be not far from 0.0015. As a matter of fact if the final ordinate had been greater by the amount 0.0018, the length of the chain from the free end to the intersection with the axis would have been in excess of 0.75. We may therefore fairly conclude that in case $\kappa = 2$ (and hence probably in all cases): *The condition $\kappa l > \frac{3}{2}$ is necessary and sufficient that the chain have a position of equilibrium other than the vertical.*

From the above discussion it appears that throughout the entire length of a curve which starts near the vertex of the parabola, the values of y' and y remain very small. As equation (5) may be written in the form

$$(5') \qquad y'' = -\frac{(\kappa y + y')(1 + y'^2)}{x + \frac{1}{2}\kappa(y_0^2 - y^2)},$$

it is further evident that at any rate when the curve reaches values of x large in comparison with y and y' — and this occurs very shortly — the difference between (5′) and

$$(5'') \qquad y'' = -\frac{(\kappa y + y')}{x}$$

becomes negligible. Of the two independent integrals of (5″) one is infinite at the origin and the other is $J_0(2\sqrt{\kappa x})$. Let us see how well

$$y = y_0 J_0(2\sqrt{\kappa x}) = y_0\left(1 - \kappa x + \frac{\kappa^2 x^2}{(1\cdot 2)^2} - \frac{\kappa^3 x^3}{(1\cdot 2\cdot 3)^2} + \cdots + (-1)^n \frac{\kappa^n x^n}{(n!)^2} + \cdots\right)$$

represents the curve of equilibrium. To begin with, the slope and radius of curvature at the origin are $-\kappa y_0$ and $2(1 + \kappa^2 y_0^2)^{3/2}/\kappa^2 y_0$. These are identical with those of the curve of equilibrium. The form of $(5'')$ shows that the inflexions of the Bessel's curve are tangent to the field. Finally, the first root of the Bessel's function is $2\sqrt{\kappa x} = 2.4+$ which in case $\kappa = 2$ becomes $0.72+$. This is remarkably close to 0.75 and represents a difference of ordinates of about 0.0013 between the curve of equilibrium and the approximation given by the Bessel's curve.

As the approximation is so extremely close in the only region where its accuracy would be under suspicion, we may safely conclude that it is safe to trust in regard to the rest of the curve. This would then establish the facts that the maximum ordinates of the curve of equilibrium approach zero as their limit. From a table of the roots of $J_0(x)$ we find

$$x = 2.40, \quad 5.52, \quad 8.65, \quad 11.79, \quad 14.93, \quad 18.07$$

for the successive roots. Hence in $J_0(2\sqrt{\kappa x})$ the successive roots are

$$\kappa x = 1.44, \quad 7.63, \quad 18.8, \quad 34.8, \quad 55.8, \quad 84.5.$$

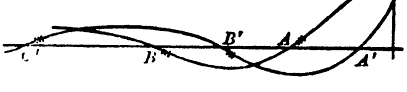
The lengthening of the successive arcs is therefore established and a value which is certainly not far from the true one is obtained. If for the purposes of calculation, we assume that these are the true values of the intersections of a curve of equilibrium with the axis when the curve lies indefinitely near the axis, we see that the minimum lengths of a chain which shall cross the axis 0, 1, 2, 3, 4, 5 times are respectively

$$1.44/\kappa, \quad 7.63/\kappa, \quad 18.8/\kappa, \quad 34.8/\kappa, \quad 55.8/\kappa, \quad 84.5/\kappa.$$

As the interval between the roots of $J_0(x)$ rapidly approaches π, it is apparent that the additional amount of chain which must be provided in order to permit an additional intersection with the axis rapidly approaches $\pi\sqrt{l/\kappa} + \pi^2/4\kappa$ where l is the amount already provided.

6. Conclusions. Suppose that the parabola $(4')$ be drawn and the equilibrium curves which start from it at each point of the contour be sketched in. It is not difficult to see what are the possible positions of equilibrium for a curve of given length. Let the length be l. If l satisfies the inequality $1\frac{1}{2} < \kappa l < a$, where a is in the neighborhood of 8, the curve will hang entirely on one side of the vertical and will depart more and more from the vertical as κl lies nearer to a and further from $1\frac{1}{2}$. If l satisfies the inequality $a < \kappa l < \beta$, where β is in the neighborhood of 19, the curve may hang entirely on one side of the vertical and in this case it will depart from the vertical more than the curves of the preceding case. Or the curve may cross the axis and will then recede from the vertical more and more as κl is nearer β and further from α. If l satisfies the inequality $\beta < \kappa l < \gamma$, where γ is in the neighborhood of 35 the curve may lie entirely on one side of the vertical and will depart from the vertical more than in the previous cases. Or it may cross the axis once and will depart further from the axis than in the preceeding case. Or it may cross the axis twice and depart from the vertical more and more as κl lies nearer γ and further from β. And so on. The discussion of the case of the chain with a free end has been pushed about as far as seems possible without actually integrating the differential equation (5).

The discussion of the case where both ends are fixed in the axis is by no means so simple. The reason is that in case there is a vertical tangent no approximate solution of (5) which shall adequately represent the curve both in the neighborhood of the vertical tangent and in the remoter regions is easily obtained. In the remoter regions, however, the approximation by means of $J_0(2\sqrt{\kappa x})$ holds also in this case. Hence the curve may be plotted in the neighborhood of the vertical tangent by the graphical method used in article 5 and then it may be pieced out with Bessel's curves. A rough sketch of the curve is given in the figure. The curve may be considered as fastened at any two of its intersections with the axis Undoubtedly there would be a definite set of inequalities to be satisfied in order that the curve should have 0, 1, 2, 3, $\cdots$ intersections with the axis (it being understood that a point of support on the axis does not count as an intersection). It is almost certain, however, that this set would not be of single entry as before, but of double entry. For the curve is double valued and each half obeys a law of decrease in ordinates and increase in length of arch similar to that found in the other case. Hence the minimum length of chain which may cut the axis once as between the points A and B' in the figure could hardly be expected to be the same as the minimum length of a chain which might cut the axis as between A' and B. In other words, if the total number of intersections were to be $n + m$, with n on one branch and m on the other, the minimum length would probably be $f(n, m)$ and not simply $f(n + m)$. But the investigation of this point would probably involve computations too laborious to be worth while.



In closing a word may be said of the behavior of the chain as the angular velocity is increased. The shape and properties of the curve are such that there is an absolute discontinuity between a possible equilibrium with different numbers of intersections with the axis. Hence if the angular velocity be gradually increased the curve cannot go gradually over from one type of equilibrium to another: the change is from one of less inclination with the axis to one of greater, and when the angular velocity has reached a certain value there will be an additional type of motion with one additional intersection with the axis but the chain will pass gradually over into it.*

In closing I should like to state that the problem of the rotating string was suggested to me, some long time since, by Professor Osgood, who had it from Professor Bôcher about a dozen years ago. The interesting feature to which my attention was called was the fact that the approximate solution by Bessel's functions was apparently valid for only a discrete set of values of κl; these are the values we have called $\kappa l = \alpha, \beta, \gamma, \cdots$. I hope that the foregoing discussion may have shown what the physical significance of this is and what the state of the chain is for intermediate values of κl.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
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* In an actual experiment with a medium brass chain about a meter long attached to an axis which could be rotated either by hand or by an electric motor, it was found difficult though not impossible to get curves which had none or one or more than five intersections with the vertical, easy to obtain those with two, three, four, or five intersections. In the first place note that extreme slowness of rotation is not particularly available owing to the large effects due to disturbances by air currents and other irregularities. Moreover if $l = 100$ and kl is about 100, which corresponds to five intersections, ω is only about five revolutions per second, which is a distinctly slow rotation. As this rotation is set up, the chain tends to climb up on itself and buckle, with the result that four or five intersections with the axis are likely to appear at the start and hence to persist. If the rotation is not much more rapid than five revolutions per second some of these intersections may easily be shaken out by sharp blows; if, however, the rotation is more rapid it becomes more and more difficult to disturb the type of motion (at any rate until such a rapidity is reached as will induce violent lateral vibrations in the chain and destroy the steadiness of the motion). This explains why the types with one or no intersections are hard to get with a rough apparatus. The difficulty in finding the types with six or more intersections appears to be due to the facts, that the chain is not a mathematical chain and that it must lie so near the axis that slight disturbances suffice to whip out one or more of the nodes. On the whole, the stability of each type of curve appears unexpectedly great.